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A New RSOS Restriction of the Zhiber-Mikhailov-Shabat Model and $\Phi_{(1,5)}$ Perturbations of Nonunitary Minimal Models

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Abstract

The RSOS restriction of the Zhiber-Mikhailov-Shabat (ZMS) model is investigated. It is shown that in addition to the usual RSOS restriction, corresponding to $\Phi_{(1,2)}$ and $\Phi_{(2,1)}$ perturbations of minimal CFT, there is another one which yields $\Phi_{(1,5)}$ perturbations of non-unitary minimal models. The new RSOS restriction is carried out and the particular case of the minimal models $\mathcal{M}_{(3,10)}$, $\mathcal{M}_{(3,14)}$ and $\mathcal{M}_{(3,16)}$ is discussed in detail. In the first two cases, while the mass spectra of the two RSOS restrictions are the same, the bootstrap systems and the detailed amplitudes are different. In the third case, even the spectra of the two RSOS restrictions are different. In addition, for $\mathcal{M}_{(3,10)}$ an interpretation in terms of the tensor product of two copies of $\mathcal{M}_{(2,5)}$ is given.

1 Introduction

It is well-known that certain perturbations of minimal models of conformal field theory (for a review see [1]) lead to massive integrable field theories [2]. In particular, the perturbations described by the operators $\Phi_{(1,2)}$, $\Phi_{(1,3)}$, $\Phi_{(2,1)}$ have this property. The description of these massive theories is given in terms of certain restrictions of other integrable field theories. Namely, the $\Phi_{(1,3)}$ perturbations are described as RSOS restrictions of sine-Gordon theory [3, 4], while the other two cases can be treated in terms of the Zhiber-Mikhailov-Shabat (ZMS) model [5, 6].

The sine-Gordon and ZMS models have quantum affine symmetry algebras $\mathcal{U}_q(A_1^{(1)})$ [7] and $\mathcal{U}_q(A_2^{(2)})$ [6], respectively. These algebras are generated by non-local conserved charges. The invariance under these quantum symmetries determines the S-matrix up to a scalar factor [5, 6, 7]. If the value of q is a root of unity, then the representation theory of the quantum group allows a consistent

truncation to a maximal spin, which is known as the RSOS restriction. In the case of sine-Gordon theory, the algebra $\mathcal{U}_q(A_1^{(1)})$ has two copies of $\mathcal{U}_q(sl(2))$ as subalgebras, both of which can be chosen to perform the restriction [3]. These two subalgebras give the same reduced theory for the reason that there is a simple automorphism of the quantum affine symmetry algebra interchanging the two subalgebras.

However, this not true for the ZMS model. $\mathcal{U}_q(A_2^{(2)})$ has the two subalgebras $\mathcal{U}_q(sl(2))$ and $\mathcal{U}_{q^4}(sl(2))$, both of which can be chosen for the restriction. The choice leading to the $\Phi_{(1,2)}$ and $\Phi_{(2,1)}$ perturbations is the first one, and this is the idea pursued in [5, 6]. In that case the fundamental particles of ZMS model form a triplet representation under the first subalgebra, which is irreducible. Under the second subalgebra the triplet decomposes as a doublet and a singlet, leaving one with the task of disentangling the amplitudes mixing these two components. Hence the restriction procedure turns out to be more complicated, but as will be shown in the sequel, it can be analysed and made systematic.

The goal of this paper is to examine the second possibility and to obtain the new RSOS restriction of the ZMS model which emerges from it. It will be argued that the S-matrices obtained in this way correspond to $\Phi_{(1,5)}$ perturbations of minimal models. The S-matrix for the case $\mathcal{M}_{(2,9)}$, which is a particular example, has been obtained by Martins in [8], where he also raised the question whether it is possible to generalise the RSOS restriction to the $\Phi_{(1,5)}$ perturbations. In the framework presented here this goal can be achieved in generality. In particular, Martins' S-matrix is reconstructed as a special case in subsection 5.1.

It is also natural to expect that there exist inequivalent RSOS restrictions for other imaginary coupling Toda field theories based on non-simply laced affine Kac-Moody algebras, too. It may be worthwhile to undertake the investigation of these theories in the future.

This article is part of a larger project devoted to the study of $\Phi_{(1,5)}$ perturbations of minimal models and their connections to $\Phi_{(1,2)}$ perturbations through the ZMS model. The other part of the work has also been completed and the paper is currently in preparation [9]. It will contain detailed analysis of the models treated in Section 5, using the method of the thermodynamical Bethe Ansatz (TBA) [10, 11, 12] and the truncated conformal space approach (TCSA) [13, 14].

The layout of the paper is the following. Section 2 gives an introduction and brief review of the S-matrix of the ZMS model, mainly in order to set up notations. Section 3 is devoted to the definition and examples of related perturbed minimal models, which are the two possible restrictions of the same ZMS model. Section 4 describes the general strategy of RSOS restriction, taking sine-Gordon theory as an example, which is then applied to the ZMS model to find the new RSOS restriction. Section 5 contains the detailed discussion of some examples. The focus will be on the $\Phi_{(1,5)}$ perturbations of the theories $\mathcal{M}_{(3,10)}$, $\mathcal{M}_{(3,14)}$ and $\mathcal{M}_{(3,16)}$, since these are the ones treated in [9]. Section 6 is reserved for the conclusions.

2 Review of the S-matrix of the ZMS model

The ZMS model is defined by the Lagrangian

$$\mathcal{L} = \int \left((\partial_\mu \phi)^2 + \frac{m^2}{\gamma^2} \left(\exp(i\sqrt{8\gamma}\phi) + \exp(-i\sqrt{2\gamma}\phi) \right) \right) dx \ , \tag{1}$$

with m being a mass parameter and γ is the coupling constant.

The model given by (1) is an imaginary coupling affine Toda theory based on the twisted affine Kac-Moody algebra $A_2^{(2)}$. This algebra is non-simply laced with two roots, corresponding to the two exponential terms in the potential. The Hamiltonian of the model is not hermitian and therefore the model is not unitary. This is in essentially different from sine-Gordon theory, which corresponds to the untwisted affine Kac-Moody algebra $A_1^{(1)}$ and has a hermitian Hamiltonian.

However, as has been shown by Smirnov [5], the model is reducible for special values of the coupling γ and the restrictions correspond to perturbations of minimal models, among them to unitary ones. More specifically, the model with $\gamma = \pi(r/s)$ can be reduced to the $\Phi_{(1,2)}$ perturbation of the minimal model $\mathcal{M}_{r,s}$ with central charge

$$c = 1 - \frac{6(r-s)^2}{rs} \ . \tag{2}$$

Therefore, while quantum field theory interpretation interpretation of the original ZMS model does not seem to be straightforward, due to its nonunitarity, we can approach the problem by taking the Lagrangian (1) to describe a putative model which can be given at least after carefully restricting it to some Hilbert space. On the other hand, there are statistical systems (e.g. the Yang-Lee edge singularity) whose description leads to nonunitary models of two-dimensional field theory and so we can hope that the models derived from (1) make sense in the realm of two-dimensional critical phenomena.

In this section I first outline the derivation of the ZMS S-matrix following [6], in order to set up conventions and fix some typos in the formulae given in [5, 6]. Then I briefly discuss the restriction carried out in [5].

2.1 The quantum symmetry of the ZMS model

Using the Lagrangian (1) it is possible to construct non-local charges commuting with the Hamiltonian. In [6] it is shown that the nonlocal charges generate the quantum affine algebra $\mathcal{A} = \mathcal{U}_q(A_2^{(2)})$ with

$$q = \exp(i\pi^2/\gamma) \ . \tag{3}$$

The defining relations of this algebra are

$$\begin{aligned} &[H_0, H_1] = 0, \\ &[H_i, E_j] = +a_{ij} E_j, \\ &[H_i, F_j] = -a_{ij} F_j, \\ &[E_i, F_j] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q_i - q_i^{-1}}, \end{aligned} \tag{4}$$

where a_{ij} is the symmetrized Cartan matrix of $A_2^{(2)}$

$$[a_{ij}] = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} \tag{5}$$

and

$$q_i = q^{a_{ii}/2} \tag{6}$$

The fundamental representation of $\mathcal{A}$ is three-dimensional and is given by the following matrices:

$$\begin{aligned}
H_0 &= \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \\
E_0 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
H_1 &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \\
E_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 0 & 0 \\ (q+q^{-1}) & 0 & 0 \\ 0 & -\left(q+q^{-1}\right) & 0 \end{bmatrix}.
\end{aligned} \tag{7}$$

The coproduct of the algebra $\mathcal{A}$ is

$$\begin{aligned}
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \\
\Delta(E_i) &= E_i \otimes q^{-H_i/2} + q^{H_i/2} \otimes E_i, \\
\Delta(F_i) &= F_i \otimes q^{-H_i/2} + q^{H_i/2} \otimes F_i,
\end{aligned} \tag{8}$$

and can be verified to be an algebra homomorphism $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. It is possible to define a counit and an antipode, but they will not be needed here.

It is important to observe that there are two copies of $\mathcal{U}_p(sl(2))$ inside the algebra $\mathcal{A}$: the generators $\{H_1,\ E_1,\ F_1\}$ form $\mathcal{U}_q(sl(2))$, while $\{H_0,\ E_0,\ F_0\}$ form $\mathcal{U}_{q'}(sl(2))$, with $q' = q^4$. These subalgebras will be denoted by $\mathcal{A}_1$ and $\mathcal{A}_0$, respectively.

The fundamental particles of the model are a triplet of kinks and the corresponding one-particle states form a so-called evaluation representation of $\mathcal{A}$, which is a rapidity-parametrized version of the fundamental representation described in (7). More precisely, we can introduce the triplet of asymptotic one-particle states as

$$|\theta, i\rangle_{as}, \tag{9}$$

where θ is the usual rapidity variable connected to the energy-momentum as $p^0 = m\cosh\theta$, $p^1 = m\sinh\theta$, with m being the particle mass and i labels the

three components. The eigenvalues of H_1 on these states give the topological charge of the kink. The label "as" can assume the values "in" and "out". The general multiparticle in-states and out-states are

$$
\begin{aligned}
&|\theta_1, i_1, \theta_2, i_2, \ldots \theta_n, i_n\rangle_{in} \\
&|\theta_n, i_n, \ldots \theta_2, i_2, \theta_1, i_1\rangle_{out},
\end{aligned}
\tag{10}
$$

with the rapidities ordered as $\theta_1 > \ldots > \theta_n$. The evaluation representation Ev_θ is defined by

$$
e_i = x_i E_i \ , \ \ f_i = x_i^{-1} F_i \ , \ \ x_i = \exp(s_i \theta), \tag{11}
$$

with $s_0 = 4\pi/\gamma - 1$, $s_1 = \pi/\gamma - 1$ (cf. [6]). This particular choice of the rapidity dependence corresponds to the so-called spin gradation. The action on the multiparticle states is easily derived using the coproduct (8). For details on quantum affine algebras and their representation theory cf. [15].

2.2 The S-matrix

The invariance of the S-matrix can be formulated as follows. The two-particle S-matrix maps the in-states to the out-states:

$$
|\theta_2, j_2, \theta_1, j_1\rangle_{out} = S_{i_1 i_2}^{j_1 j_2}(\theta_1 - \theta_2) |\theta_1, i_1, \theta_2, i_2, \rangle_{in} \tag{12}
$$

We require that the two-particle S-matrix must be invariant under the action of the quantum affine symmetry algebra on the asymptotic states. Let us define the matrix

$$
R(x, q) = P_{12} \hat{S} \ , \tag{13}
$$

where P_{12} is the permutation operator acting on the internal indices of the two-particle state and $\hat{S}$ denotes the tensor part of the S-matrix. The quantum symmetry specifies the S-matrix only up to a scalar function - the precise normalization of the tensor part will be clear from (16) below. We also change the gradation to the so-called homogeneous one, in which the complete rapidity dependence is carried by E_0 and F_0:

$$
e_0^h = x E_0 \ , \ \ f_0^h = x^{-1} F_0 \ , \ \ e_1^h = E_1 \ , \ \ f_1^h = F_1 \ , \ \ x = x_0 x_1^2 \ . \tag{14}
$$

This can be implemented by the following algebra automorphism:

$$
\mathcal{A} \to \mathcal{A} \ , \ \ a \mapsto x_1^{-H_1/2} a x_1^{H_1/2}. \tag{15}
$$

The invariance of the S-matrix implies that $R(x, q)$ has to intertwine between the representations $Ev_{\theta_1} \otimes Ev_{\theta_2}$ and $Ev_{\theta_2} \otimes Ev_{\theta_1}$. With the variable x defined in terms of the relative rapidity $\theta = \theta_1 - \theta_2$, $R(x, q)$ is given by

$$
R(x, q) = \frac{(1 - q^4)(q^6 + 1)}{q^5} P_{12} + \frac{x - 1}{q^3} R_{12} + \frac{q^3(x - 1)}{x} R_{21}^{-1}, \tag{16}
$$

where

$$R_{12} = \begin{bmatrix} q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{q^4-1}{q^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^2 & 0 & \frac{q^4-1}{q} & 0 & \frac{-q^4+q^6-q^2+1}{q^2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{q^4-1}{q} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{q^4-1}{q^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} \end{bmatrix} . \quad (17)$$

It plays a central role in the investigation of the ZMS model in the framework of quantum inverse method and was originally obtained by Izergin and Korepin in that context [17].

$R(x,q)$ is a solution of the parametric quantum Yang-Baxter equation:

$$R_{12}(x/y,q)R_{13}(x,q)R_{23}(y,q) = R_{23}(y,q)R_{13}(x,q)R_{12}(x/y,q) \ . \quad (18)$$

and satisfies the following important relations, written in terms of $\hat{S}(x)$:

$$\begin{aligned} \hat{S}(x)\hat{S}(1/x) &= \frac{\left(x+q^6\right)\left(x-q^4\right)\left(1+q^6x\right)\left(1-xq^4\right)}{q^{10}x^2}, \\ \hat{S}(-q^6/x) &= (1\otimes\alpha)\hat{S}_{21}^{t_1}(x,q)(1\otimes\alpha), \end{aligned} \quad (19)$$

with

$$\alpha = \begin{bmatrix} 0 & 0 & q^{-1} \\ 0 & 1 & 0 \\ q & 0 & 0 \end{bmatrix} . \quad (20)$$

The first property in (19) ensures "unitarity", while the second means that in the spin gradation the S-matrix is actually crossing symmetric. The term "unitarity" must be understood carefully: here we refer to the property of the S-matrix given by

$$S_{ij}^{kl}(-\theta)S_{kl}^{mn}(\theta) = \delta_i^m\delta_j^n \ , \quad (21)$$

while the quantum field theory itself is not unitary, which can be seen e.g. from the negative sign of the residues at the poles of the S-matrices of singlet bound states in the theory.

The solution for the S-matrix is fixed by its intertwining property up to multiplication with a scalar function. To make it unitary, while preserving crossing symmetry, it has to be multiplied by the following factor [5] (with an overall sign that can be chosen arbitrarily):

$$\begin{aligned} S_0(\theta) = \pm\frac{1}{4i}&\left(\sinh\frac{\pi}{\xi}(\theta-\pi i)\sinh\frac{\pi}{\xi}\left(\theta-\frac{2\pi i}{3}\right)\right)^{-1}\times \\ &\exp\left(-2i\int_0^\infty\frac{\sin k\theta\sinh\frac{\pi k}{3}\cosh\left(\frac{\pi}{6}-\frac{\xi}{2}\right)k}{k\cosh\frac{\pi k}{2}\sinh\frac{\xi k}{2}}dk\right) \ , \end{aligned} \quad (22)$$

with ξ given by

$$\xi = \frac{2}{3}\left(\frac{\pi\gamma}{2\pi-\gamma}\right) \ . \tag{23}$$

Using ξ, the variable x can be alternatively written as

$$x = \exp\left(\frac{2\pi\theta}{\xi}\right) . \tag{24}$$

In the forthcoming paper [9], where the S-matrices proposed later in this work are put to some stringent tests, it will be shown that the sign choice in (22) is tied up with the statistics of the particle and does not have any physical meaning in itself (in two dimensions, statistics has no inherent meaning either, only in conjunction with the choice of whether $S(\theta=0)$ is $+1$ or -1, see in the context of the thermodynamical Bethe Ansatz, e.g. [10]).

The S-matrix given above is just the S-matrix of the lowest lying state, which is the fundamental kink triplet. If one needs to have the complete S-matrix, it can be completed using the usual methods of the S-matrix bootstrap. In this way, generically, higher kinks and breathers arise. The fundamental kink S-matrix has poles at $\theta = i\pi - i\xi m$ and $\theta = 2i\pi/3 - i\xi m$, with m being an integer. The ones in the physical strip $0 \leq \Im m\theta < \pi$ correspond to bound states. The first sequence corresponds to breathers. At these points $\hat{S}(x)$ degenerates into a rank-one projector, indicating that there is a singlet bound state. The second sequence corresponds to higher kinks, since there $\hat{S}(x)$ degenerates into a rank-three projector, corresponding to a triplet of particles. All poles have their crossing symmetric counterparts at $\theta = i\xi m$ and $\theta = i\pi/3 + i\xi m$, respectively. The breather-kink and breather-breather S-matrices are all scalars, while the kink-kink S-matrices have the following general form:

$$\hat{S}\left(\exp\left(\frac{2\pi}{\xi}\theta + i\phi(k_1,k_2)\right)\right) S_0^{k_1k_2}(\theta), \tag{25}$$

as shown in [5]. Here $k_1,\ k_2$ number the kinks, $S_0^{k_1k_2}(\theta)$ is a scalar function similar to $S_0(\theta)$ and $\phi(k_1,k_2)$ is a phase shift satisfying $\phi(k_1,k_2) = -\phi(k_2,k_1)$. I wish to note, however, that the formula given in [5] for the explicit expression of the tensor part of the kink-higher kink S-matrix (eqn. (3.7) in [5]) is off by a scalar factor. The correct version is

$$A\mathcal{P}_{12}R_{13}(xq^2,q)R_{23}(x/q^2,q)\mathcal{P}_{12}A^{-1} = \frac{(1-xq^2)(x+q^8)}{q^5x}R_{(12)3}(-x,q), \tag{26}$$

where $\mathcal{P}_{12}$ is the projector on the spin-1 representation in the product of two spin-1 representations, the index (12) denotes its 3-dimensional image, and A denotes a redefinition of the states given by the matrix

$$\begin{bmatrix} I_3 & 0 & 0 \\ 0 & -\frac{-4\,q^2+q^4+1}{2\,q^2}I_3 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{27}$$

(with I_3 representing the 3×3 unit matrix), which is equivalent to redefining the middle component of the higher kink triplet.

Note that the fundamental distinction between a kink and a breather is given by the fact that kinks are not singlets under the quantum algebra, they come in triplets, however, breathers are singlets. This essentially amounts to saying that while kinks carry topological charge, the breathers do not, since the topological charge is part of the quantum algebra.

2.3 RSOS restriction of ZMS model and $\Phi_{(1,2)}$ perturbations of minimal models

The RSOS restriction, described in [5, 6], proceeds in the following way. If q is a root of unity, then we know that the representation of the quantum algebra is different from the case of generic q. Supposing that

$$\gamma = \pi(r/s) \ , \tag{28}$$

we get $q^r = \pm 1$. In this case it is possible to consistently truncate the Hilbert space to representations of $\mathcal{A}_1$, which have spins not exceeding $j_{max} = (r-2)/2$. The details of the construction are given in numerous papers in the context of sine-Gordon theory (see e.g. [3, 4]) as well as the ZMS model (e.g. [5, 6]) and it will be outlined in the section dealing with the other RSOS restriction of the ZMS model.

The truncation of tensor product is the same which occurs in minimal models of conformal field theory (see [1]; for a detailed exposition of the quantum group structure of minimal models, see [18] and references therein). Indeed, the models obtained in this way can be considered as perturbations of minimal models $\mathcal{M}_{r,s}$ with the field $\Phi_{(1,2)}$ in the Kac table. This is known to be a relevant and integrable perturbation of all minimal models [2].

The way, in which this identification arises, is crucial to the considerations of this paper. As shown in [19, 20] the minimal model $\mathcal{M}_{r,s}$ can be considered as the conformal quantization of the imaginary coupling Liouville field theory given by

$$\mathcal{L} = \int ((\partial_\mu \phi)^2 + \exp(i\sqrt{8\gamma}\phi))dx \ . \tag{29}$$

The operators

$$\exp\left(-i\frac{n-1}{2}\sqrt{8\gamma}\phi\right) \tag{30}$$

can be identified with the primary fields $\Phi_{(1,n)}$. It is then easy to see the correspondence between the ZMS model at the coupling $\gamma = \pi(r/s)$ and the $\Phi_{(1,2)}$ perturbation of the minimal model $\mathcal{M}_{r,s}$.

3 Relation to $\Phi_{(1,5)}$ perturbations

The identification of ZMS model and perturbed minimal models, described at the end of the previous section, lends itself to the following idea. Interchanging the role of the exponentials in (1), i.e. taking the second one as the part of the imaginary coupling Liouville model and the first one as perturbation, one arrives at another integrable model, which on the level of the ZMS model is

the same as the starting point. The question is: what happens after RSOS restriction?

In sine-Gordon theory, where the two exponentials have the same exponents with opposite signs, this interchange of the two terms can be compensated by a simple automorphism of the quantum symmetry algebra. Here this is not the case, because the two roots have different lengths. Generically, such reshuffling is expected to yield different models after RSOS restriction for any imaginary coupling Toda field theory associated to a non-simply laced quantum affine algebra.

It can be easily checked, using the imaginary coupling Liouville theory description, that the new model will correspond to a $\Phi_{(1,5)}$ perturbation of another minimal model $\mathcal{M}_{r',s'}$ with

$$\frac{r'}{s'} = \frac{1}{4}\frac{r}{s} \ . \tag{31}$$

The $\Phi_{(1,5)}$ perturbation of a unitary minimal model is irrelevant. Hence we can expect only non-unitary cases to be interesting. Some examples:

- The model $\mathcal{M}_{8,9}+\Phi_{(1,2)}$ is related to the model $\mathcal{M}_{2,9}+\Phi_{(1,5)}$, which is the same (due to the symmetry of the Kac table) as the model $\mathcal{M}_{2,9}+\Phi_{(1,4)}$. This correspondence was used to prove the integrability and to conjecture the S-matrix of this particular case by Martins et al. [8, 21].

- The case of the magnetic perturbation of the Ising model, i.e. $\mathcal{M}_{3,4}$. It is known to have an E_8 S-matrix. The related model is the $\Phi_{(1,5)}$ perturbation of $\mathcal{M}_{3,16}$.

- The $\Phi_{(2,1)}$ perturbation of $\mathcal{M}_{5,6}$, which can be thought of as the $\Phi_{(1,2)}$ perturbation of $\mathcal{M}_{6,5}$. It is related by the above correspondence to the $\Phi_{(1,5)}$ perturbation of $\mathcal{M}_{3,10}$.

- The $\Phi_{(1,2)}$ perturbation of $\mathcal{M}_{6,7}$ gets mapped into the $\Phi_{(1,5)}$ perturbation of $\mathcal{M}_{3,14}$.

In this paper the focus will be on the models $\mathcal{M}_{3,10}$, $\mathcal{M}_{3,14}$ and $\mathcal{M}_{3,16}$. These are the models which are subjected to the TCSA and TBA analysis in [9].

4 The RSOS restriction of the ZMS model

4.1 RSOS restriction

First I briefly recall some necessary facts about RSOS restriction, which will prove useful later in the study of the ZMS case. A more detailed exposition can be found in e.g. [4].Take a doublet of solitons, transforming under $\mathcal{U}_q(sl(2))$. The tensorial part of the S-matrix will be a linear combination of the following matrix

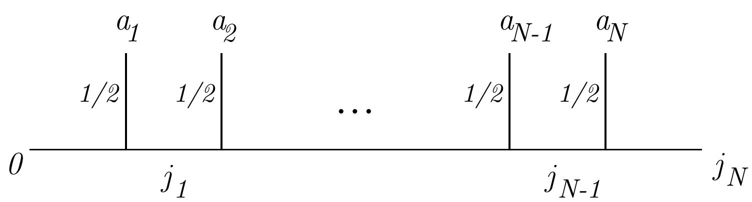
$$\hat{\mathcal{R}} = \sqrt{q}P_{12}\mathcal{R}^{\frac{1}{2}\frac{1}{2}} = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q-1/q & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \tag{32}$$

and its inverse.

The many-particle Hilbert space can be decomposed into irreducible representations of $\mathcal{U}_q(sl(2))$:

$$\mathcal{V}_{\frac{1}{2}}^{\otimes N} = \bigoplus_{\frac{1}{2}, j_2, \ldots j_N} \mathcal{V}_{j_N} \ , \tag{33}$$

where $\mathcal{V}_j$ is the spin-j representation of the quantum group and $j_{i+1} = j_i \pm \frac{1}{2} \geq 0$ are the intermediate representations in the N-fold tensor product. It can be represented graphically as



(34)

In this way we introduce a new labelling of the multiparticle Hilbert space. Instead of identifying the states by the topological charge, which is the eigenvalue of H_0, we decompose the space with respect to the intermediate representation of the tensor product.

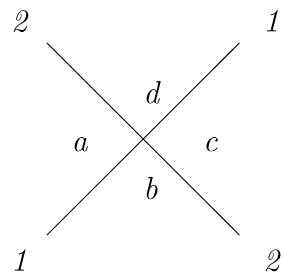
The RSOS restriction amounts to truncating the Hilbert space up to the spin $j_{max} = p/2 - 1$, where p is the first positive integer for which $q^p = \pm 1$. It corresponds to the truncation of the algebra $\mathcal{U}_q(sl(2))$ to $\mathcal{U}_q^{res}(sl(2))$ in the notations [15]. In terms of the representation labels, there are only finitely many sectors left in the Hilbert space. The sine-Gordon S-matrix maps the truncated space back unto itself, so it is consistent to take the restriction.

Some care must be taken about the Hilbert space structure in the restricted space. The original theory has a Hilbert space of its own, but the restriction eliminates part of it. Hence it is necessary to introduce a new inner product on the restricted space. The new product, however, turns out to define a unitary theory only in special cases, when a Hermitian structure can be found. That is possible e.g. in the examples corresponding to perturbations of the unitary series of minimal models. This is described in [3], to which the interested reader is referred for more details. In the ZMS case, the model is not unitary to start with, however, after the RSOS restriction, it is still possible to end up with unitary theories [5].

The doublet solitons yield kinks going back and forth between the sectors. A kink K_{ab} is labelled by the two representations a and b between which it interpolates. The possible scattering processes look like

$$K_{ab}(\theta_1) + K_{bc}(\theta_2) \rightarrow K_{ad}(\theta_2) + K_{dc}(\theta_1) \tag{35}$$

In graphical terms this process is conventionally represented as



The lines are the world-lines of the kinks, with their rapidities assigned, while the spaces between the lines are indexed according to the sectors between which the kinks interpolate.

One finds that to describe the amplitude of the above process the following substitutions have to be made in the S-matrix [4]:

$$\begin{aligned}\hat{\mathcal{R}} &\to \sqrt{q}(-1)^{d+b-a-c}q^{C_a+C_c-C_b-C_d}\left\{\begin{array}{ccc}\frac{1}{2} & a & b\\ \frac{1}{2} & c & d\end{array}\right\}_q,\\ \hat{\mathcal{R}}^{-1} &\to (\sqrt{q})^{-1}(-1)^{d+b-a-c}q^{C_b+C_d-C_a-C_c}\left\{\begin{array}{ccc}\frac{1}{2} & a & b\\ \frac{1}{2} & c & d\end{array}\right\}_q,\end{aligned} \tag{36}$$

where $C_a = a(a+1)$.

The restriction procedure can be appropriately generalized to the case when the fundamental representation is spin-1 instead of spin-1/2 [5]. The breather-soliton amplitudes and the breather-breather amplitudes remain the same, since the breathers are singlets (cf. [3]). It has been applied to the sine-Gordon models to get S-matrices for $\Phi_{(1,3)}$-perturbed minimal models [3, 4] and the spin-1 generalization leads to the S-matrices of the $\Phi_{(1,2)}$-perturbations using the ZMS model as starting point [5, 6].

4.2 The new RSOS restriction of the ZMS model

Consider now the RSOS restriction of the ZMS model with respect to the second quantum group $\mathcal{A}_0 = \mathcal{U}_{q^4}(sl(2))$. The fundamental triplet splits into a doublet and a singlet under the action of $\mathcal{A}_0$. This is where the subtlety of the procedure lies. Since the representation is reducible, there are possible mixing terms. The kink triplet splits into a doublet of what can be called charged kinks, and a singlet neutral kink. Now one may try to argue that since the neutral kink is a singlet, it has nothing to do with the restriction procedure, like the breathers. However, this state is degenerate in mass with the charged kinks. Therefore there are amplitudes which describe the fusion of two charged kinks into a neutral one and vice versa, which is essentially different from the properties of the breathers.

It is necessary to go over to a new gradation in which the rapidity dependence is taken over by the generators in $\mathcal{A}_1$. It will be convenient to use the variable

$$y = \sqrt{x} \tag{37}$$

instead of x. In this gradation, the quantum symmetry of the S-matrix is described by the following intertwining equations:

$$\begin{aligned}
&\tilde{R}(y,q)\,(H_i\otimes 1+1\otimes H_i)=(H_i\otimes 1+1\otimes H_i)\,\tilde{R}(y,q),\\
&\tilde{R}(y,q)\left(E_1\otimes q^{-H_1/2}+yq^{H_1/2}\otimes E_1\right)=\left(E_1\otimes q^{H_1/2}+yq^{-H_1/2}\otimes E_1\right)\tilde{R}(y,q),\\
&\tilde{R}(y,q)\left(F_1\otimes q^{-H_1/2}+\frac{1}{y}q^{H_1/2}\otimes F_1\right)=\left(F_1\otimes q^{H_1/2}+\frac{1}{y}q^{-H_1/2}\otimes F_1\right)\tilde{R}(y,q),\\
&\tilde{R}(y,q)\left(E_0\otimes q^{-H_0/2}+q^{H_0/2}\otimes E_0\right)=\left(E_0\otimes q^{H_0/2}+q^{-H_0/2}\otimes E_0\right)\tilde{R}(y,q),\\
&\tilde{R}(y,q)\left(F_0\otimes q^{-H_0/2}+q^{H_0/2}\otimes F_0\right)=\left(F_0\otimes q^{H_0/2}+q^{-H_0/2}\otimes F_0\right)\tilde{R}(y,q).
\end{aligned}\tag{38}$$

The tensor part of the S-matrix is given by

$$\mathcal{S}=P_{12}\tilde{R}(y,q)\ ,\tag{39}$$

and has the following nonzero matrix elements:

$$\begin{aligned}
&\mathcal{S}_{++}^{++}(y,q)=\mathcal{S}_{--}^{--}(y,q)=\frac{(y^2-q^4)(y^2+q^6)}{y^2q^5}\\
&\mathcal{S}_{+0}^{+0}(y,q)=\mathcal{S}_{-0}^{-0}(y,q)=\mathcal{S}_{0+}^{0+}(y,q)=\mathcal{S}_{0-}^{0-}(y,q)=-\frac{(q^4-1)(y^2+q^6)}{yq^5}\\
&\mathcal{S}_{+0}^{0+}(y,q)=\mathcal{S}_{-0}^{0-}(y,q)=\mathcal{S}_{0+}^{+0}(y,q)=\mathcal{S}_{0-}^{-0}(y,q)=\frac{(y^2+q^6)(y^2-1)}{y^2q^3}\\
&\mathcal{S}_{+-}^{-+}(y,q)=\mathcal{S}_{-+}^{+-}(y,q)=\frac{(y^2-1)(y^2+q^2)}{y^2q}\\
&\mathcal{S}_{+-}^{+-}(y,q)=-\frac{(q^4-1)(q^6+q^4y^2-q^4+y^2)}{q^5}\\
&\mathcal{S}_{-+}^{-+}(y,q)=-\frac{(q^4-1)(q^6+y^2-q^2+y^2q^2)}{q^5y^2}\\
&\mathcal{S}_{+-}^{00}=\mathcal{S}_{00}^{+-}=-q^4\mathcal{S}_{-+}^{00}=-q^4\mathcal{S}_{00}^{-+}=-\frac{(q^4-1)(y^2-1)}{y}\\
&\mathcal{S}_{00}^{00}=\frac{q^6y^2+y^2q^8-q^8-q^4y^2+y^2-q^{10}y^2+y^4q^2-y^2q^2}{y^2q^5}
\end{aligned}\tag{40}$$

This matrix $\tilde{R}(y,q)$ can, in fact, be obtained from the matrix $R(x,q)$ in (16) by a similarity transformation analogous to (15), hence it satisfies the same unitarity relation and an appropriate crossing condition with a redefined α (see (19)).

Proceeding to the RSOS restriction, let us first examine the charged kink part. It is a 4×4 submatrix of $\mathcal{S}$, which can be expressed in terms of the fundamental R-matrix of $\mathcal{U}_{q^4}(sl(2))$, analogously to the case of sine-Gordon theory. This yields

$$\left(\frac{y^2}{q}-\frac{(1-q^2)q^3}{1+q^4}\right)\hat{\mathcal{R}}\left(q^4\right)^{-1}-\left(\frac{q}{y^2}-\frac{1-q^2}{q(1+q^4)}\right)\hat{\mathcal{R}}\left(q^4\right)\ .\tag{41}$$

Here

$$\hat{\mathcal{R}}\left(q^4\right) \tag{42}$$

denotes the matrix obtained from $\hat{\mathcal{R}}$, given in (32), by the substitution $q \to q^4$.

The prescription given in (36) yields the following result, using the explicit expression of the $6-j$ symbols given in the Appendix A of [4]:

$$\begin{array}{ccc} & d & \\ a & \times & c \\ & b & \end{array} = \left(\frac{y^2}{q} - \frac{q}{y^2} - \frac{1}{q} + q\right)\delta_{ac}\left(\frac{[2b+1][2d+1]}{[2a+1][2c+1]}\right)^{1/2} + \left(\frac{y^2}{q^5} - \frac{q^5}{y^2} - \frac{1}{q} + q\right)\delta_{bd} \tag{43}$$

for the charged kink scattering. The q-numbers in this equation are defined with respect to q^4:

$$[x] = \frac{q^{4x} - q^{-4x}}{q^4 - q^{-4}} \ . \tag{44}$$

For the diagram to give a non-zero result, neighbouring labels a, b, c, d should differ by $\pm 1/2$. This amplitude is not crossing symmetric, but needs a similarity transformation to achieve crossing symmetry, like the original S-matrix. The transformation is performed by multiplying the amplitude with (c.f. [4])

$$\left(\frac{[2b+1][2d+1]}{[2a+1][2c+1]}\right)^{-\theta/2\pi i} \ . \tag{45}$$

The next step is to examine what happens to the neutral kink. Its presence means that it is possible to take two neighbouring vacua to be the same. The following additional processes are allowed:

- Neutral kink scattering, neutral kink-charged kink forward scattering and neutral kink-charged kink reflecion, respectively. These are simple, since there are no associated group structures.

$$\begin{array}{ccc} & a & \\ a & \times & a \\ & a & \end{array} = \mathcal{S}^{00}_{00}(\theta) \qquad \begin{array}{ccc} & b & \\ a & \times & b \\ & a & \end{array} = \mathcal{S}^{+0}_{0+}(\theta) \qquad \begin{array}{ccc} & a & \\ a & \times & b \\ & a & \end{array} = \mathcal{S}^{+0}_{+0}(\theta) \tag{46}$$

- Two charged kinks turn into two neutral kinks. In this channel, there is a Clebsh-Gordan coefficient to deal with. The presence of this coefficient is responsible for the fact $\mathcal{S}^{00}_{+-} = -q^4\mathcal{S}^{00}_{-+}$. This means that the transition amplitude from the spin-1 state of the two charged kinks to the two neutral kinks vanishes:

$$\frac{1}{q^2}\mathcal{S}^{00}_{+-} + q^2\mathcal{S}^{00}_{-+} = 0. \tag{47}$$

The amplitude for the process is given by the singlet component and turns out to be

$$\begin{array}{ccc} & a & \\ a & \times & a \\ & b & \end{array} = i\frac{(q^4-1)(y^2-1)}{q^2 y} \tag{48}$$

The factor i is necessary to achieve crossing symmetry of the result. (The crossing symmetry transformation in the variable y takes the form $y \to iq^3/y$, as can be seen from (19)).

- Two neutral kinks turn into two charged kinks. Since the unreduced S-matrix is time-reflection symmetric, this amplitude is the same as the one above.

This concludes the description of the reduced S-matrix.

5 Some explicit examples

5.1 The model $\mathcal{M}_{(2,9)} + \Phi_{(1,5)}$

The simplest models to consider are $\mathcal{M}_{(2,n)}$. In that case $q'^2 = 1$, which means that the maximal spin allowed is 0. Hence charged kinks are frozen. The models contain only the neutral kink and the breathers. The related models are $\Phi_{(1,2)}$ perturbations of the models $\mathcal{M}_{(8,n)}$. The particular case of the S-matrix of the model $\mathcal{M}_{(2,9)} + \Phi_{(1,5)}$, which was derived and tested using the TBA and TCSA approach by Martins et al. [8, 21], can also be obtained in the framework presented here. Using the matrix element $\mathcal{S}_{00}^{00}$ in (40), after some computation the amplitude reduces to the form

$$-4i \sinh \frac{\pi}{\xi} (\theta + i\pi) \sinh \frac{\pi}{\xi} \left(\theta + \frac{2i\pi}{3} \right) S_0(\theta) \ , \tag{49}$$

where $\xi = 8\pi/15$. From [21] (formula (A.7)) one has

$$\begin{aligned} S_0(\theta) = -\frac{1}{4i} \left(\sinh \frac{\pi}{\xi} (\theta + i\pi) \sinh \frac{\pi}{\xi} \left(\theta + \frac{2i\pi}{3} \right) \right)^{-1} \times \\ f_{2/3}(\theta) f_{2/15}(\theta) f_{7/15}(\theta) f_{-1/15}(\theta) f_{-2/5}(\theta) \ , \end{aligned} \tag{50}$$

with the notation

$$f_x(\theta) = \frac{\tanh \frac{1}{2}(\theta + ix\pi)}{\tanh \frac{1}{2}(\theta - ix\pi)} \ . \tag{51}$$

The final result for the amplitude is

$$f_{2/3}(\theta) f_{2/15}(\theta) f_{7/15}(\theta) f_{-1/15}(\theta) f_{-2/5}(\theta) \ , \tag{52}$$

which agrees with [8, 21].

In [21], this amplitude was shown to correspond to a particular combination of the amplitudes of $\mathcal{M}_{(8,9)} + \Phi_{(1,2)}$, but the precise connection remained obscure. The present derivation shows that this special combination is just the amplitude of the neutral kink in the unreduced theory, which is (by virtue of the RSOS restriction framework) the S-matrix obtained after restriction.

More interesting for us are the models $\mathcal{M}_{(3,2n)}$, since $\mathcal{M}_{(3,10)}$, $\mathcal{M}_{(3,14)}$ and $\mathcal{M}_{(3,16)}$, which are treated in detail in [9], fall into this class.

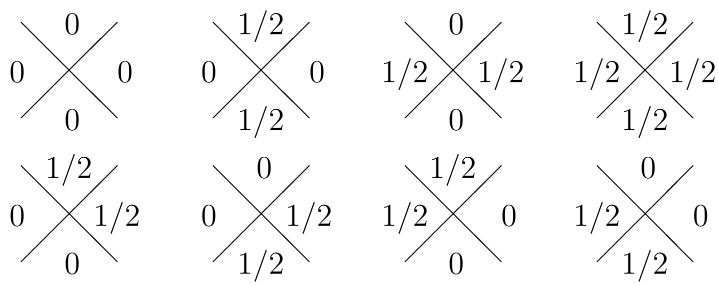
5.2 The RSOS amplitudes for $\mathcal{M}_{(3,10)}$

The minimal model $\mathcal{M}_{(3,10)}$ has central charge $c = -44/5$. The Kac table consists of two rows and contains 9 different conformal primary fields (which can be taken e.g. as the elements of the first row), together with the identity. The field $\Phi_{(1,5)}$ has scaling dimension $-2/5$, hence it generates a relevant perturbation of the model.

The S-matrix can be evaluated using the formulas in the previous section. The related unitary model is $\mathcal{M}_{(5,6)} + \Phi_{(2,1)}$, which corresponds to

$$q = \exp\left(i\pi\frac{5}{6}\right)\ ,\ q' = q^4 = \exp\left(i\pi\frac{10}{3}\right)\ . \tag{53}$$

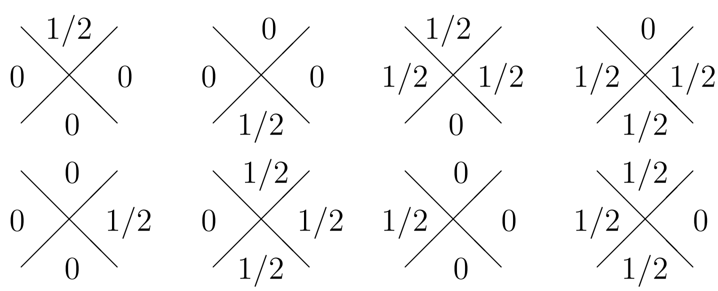
Since $q'^3 = 1$, the maximum allowed spin is $1/2$. Hence charged kinks are allowed and direct computation shows that there are only two independent amplitudes. The eight amplitudes



are equal to

$$\frac{-i(y^2-1)^2}{y^2}\ , \tag{54}$$

while the other eight



are given by

$$\frac{\sqrt{3}(y^2-1)}{y}\ . \tag{55}$$

For this model

$$\xi = \pi\ ,\ y = \exp\left(\frac{\pi}{\xi}\theta\right) = \exp(\theta)\ , \tag{56}$$

where $\theta = \theta_1 - \theta_2$ is the relative rapidity of the particles. The amplitudes can be arranged into an 8×8 matrix, with the rows and columns indexed by sequences $\{j_1, j_2, j_3\},\ j_i = 0, 1/2$. These sequences indicate the vacua between which

the particles in the two-particle states mediate. The only possibly non-zero matrix-elements are obtained only if the first and third label of the incoming and outgoing two-particle states agrees, which means that the matrix is block-diagonal with 2×2 blocks. In this form, it is easy to check the unitarity of the transition amplitude.

5.3 Particle interpretation

The above S-matrix is in a "kink picture", which has the unusual feature that not all possible sequences of kinks are allowed, only the ones in which neighbouring kinks share the same vacuum states. Note that under the interchange of the vacua $0 \leftrightarrow 1/2$, which is a Z_2 map, the amplitudes remain unchanged. Hence we can choose to make an identification of the RSOS sequences in the following way:

$$\{j_1, j_2, \ldots j_n\} \equiv \{1/2 - j_1, 1/2 - j_2, \ldots 1/2 - j_n\} \ . \tag{57}$$

Recalling that the multiparticle state with $n-1$ particles corresponding to an RSOS sequence $\{j_1, j_2, \ldots j_n\}$ is given by:

$$K_{j_1 j_2}(\theta_1) K_{j_2 j_3}(\theta_2) \ldots K_{j_{n-1} j_n}(\theta_{n-1}) \ , \tag{58}$$

it is clear that the result is the identification $K_{0,0} \equiv K_{1/2,1/2}$ and $K_{0,1/2} \equiv K_{1/2,0}$. Let us call K the particle obtained from the first pair, and $\tilde{K}$ the one coming from the second. Then the scattering matrix reduces to a four by four matrix describing the scattering of K and $\tilde{K}$. Their S-matrix reads as follows:

$$\begin{aligned}
S_{KK}^{KK}(\theta) &= S_{\tilde{K}\tilde{K}}^{\tilde{K}\tilde{K}}(\theta) = S_{\tilde{K}K}^{\tilde{K}K}(\theta) = S_{K\tilde{K}}^{K\tilde{K}}(\theta) = -4i \sinh^2(\theta) S_0(\theta) \\
S_{KK}^{\tilde{K}\tilde{K}}(\theta) &= S_{\tilde{K}\tilde{K}}^{KK}(\theta) = S_{\tilde{K}K}^{K\tilde{K}}(\theta) = S_{K\tilde{K}}^{\tilde{K}K}(\theta) = 4 \sinh(\theta) \sin\left(\frac{2\pi}{3}\right) S_0(\theta)
\end{aligned} \tag{59}$$

Using the identity

$$\frac{\sinh \frac{\pi}{\zeta}\left(\theta + i\alpha \frac{2}{3}\pi\right)}{\sinh \frac{\pi}{\zeta}\left(\theta - i\alpha \frac{2}{3}\pi\right)} = \exp\left(-i \int_{-\infty}^{\infty} \frac{dk}{k} \sin k\theta \frac{\sinh\left(\frac{1}{2}\zeta - \alpha \frac{2}{3}\pi\right) k}{\sinh \frac{1}{2} k\zeta}\right) \tag{60}$$

and choosing the positive sign in (22), the function $S_0(\theta)$ can be calculated with the result

$$-\frac{1}{4i}\left(\sinh(\theta) \sinh\left(\theta - \frac{2\pi i}{3}\right)\right)^{-1} \frac{\sinh \frac{1}{2}\left(\theta + \frac{\pi i}{3}\right)}{\sinh \frac{1}{2}\left(\theta - \frac{\pi i}{3}\right)} \ . \tag{61}$$

The eigenvalues of the two-particle S-matrix are

$$1, \quad \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \ , \tag{62}$$

using the following notation common in the context of the S-matrix bootstrap

$$(p) = \frac{\sinh\left(\frac{\theta}{2} + p\frac{\pi i}{2}\right)}{\sinh\left(\frac{\theta}{2} - p\frac{\pi i}{2}\right)} \ . \tag{63}$$

The two-particle states on which the scattering is diagonal, are just symmetric and antisymmetric combinations of the two-particle states, in terms of the particles K and $\tilde{K}$:

$$|K(\theta_1)K(\theta_2)\rangle \pm |\tilde{K}(\theta_1)\tilde{K}(\theta_2)\rangle \ , \ |K(\theta_1)\tilde{K}(\theta_2)\rangle \pm |\tilde{K}(\theta_1)K(\theta_2)\rangle \ . \tag{64}$$

The eigenvalue 1 corresponds to the antisymmetric, while the other eigenvalue to the symmetric combinations. It is possible to define new one-particle states as follows:

$$|A\rangle = (|K\rangle + |\tilde{K}\rangle)/\sqrt{2} \ , \ |B\rangle = (|K\rangle - |\tilde{K}\rangle)/\sqrt{2} \ . \tag{65}$$

We introduce a charge conjugation under which A and B are selfconjugate particles. This is consistent, since the eigenvalues above are transformed into themselves under crossing symmetry.

The S-matrix in this basis is particularly simple, and the scattering reduces to diagonal form:

$$S_{AA} = S_{BB} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \ , \ S_{AB} = 1 \ . \tag{66}$$

Note that S_{AA} and S_{BB} are just two copies of the S-matrix of the model $\mathcal{M}_{(2,5)} + \Phi_{(1,2)}$, in which the spectrum consists of a self-conjugate scalar particle. This is not surprising if one notes that there is a different way of thinking about $\mathcal{M}_{(3,10)}$, namely, as the tensor product $\mathcal{M}_{(2,5)} \otimes \mathcal{M}_{(2,5)}$. The model $\mathcal{M}_{(2,5)}$ has central charge $c = -22/5$ and contains two conformal families: one of them is given by the identity, the other one is generated by the operator $\phi_{(1,2)}$ with dimension $-2/5$.

$\mathcal{M}_{(2,5)} \otimes \mathcal{M}_{(2,5)}$ contains two Virasoro algebras given by the operators

$$1 \otimes T(z) \ , \ T(z) \otimes 1 \ , \tag{67}$$

where $T(z)$ is the energy-momentum tensor in $\mathcal{M}_{(2,5)}$. Their symmetric combination is the energy-momentum tensor in $\mathcal{M}_{(3,10)}$. The fields in $\mathcal{M}_{(3,10)}$ can be classified into Z_2-even and Z_2-odd fields with respect to the Z_2 map provided by flipping the tensor product. The fields with Kac label $(1,n)$ in the $\mathcal{M}_{(3,10)}$ model give Z_2-even fields when n is odd and Z_2-odd fields when n is even. The two sectors are the even and odd sector, respectively. The perturbing operator $\Phi_{(1,5)}$ is in the even sector and is nothing other than $(1 \otimes \phi_{(1,2)} + \phi_{(1,2)} \otimes 1)/\sqrt{2}$. The above results clearly reflect this correspondence.

The S-matrix of $\mathcal{M}_{(2,5)} + \Phi_{(1,2)}$ has the ϕ^3 property, i.e. the particle occurs as bound state of itself. This property is equally valid at the level of the $\mathcal{M}_{(3,10)}$ model. The residue at the bound state pole has wrong sign, which reflects the nonunitarity of the theory.

5.4 The model $\mathcal{M}_{(3,14)} + \Phi_{(1,5)}$

Now let us turn to the model $\mathcal{M}_{(3,14)} + \Phi_{(1,5)}$. The reduced amplitudes turn out to be identical to (54,55), up to some changes in the sign. Now

$$\xi = \pi/2 \ , \ y = \exp(2\theta) \ . \tag{68}$$

The S-matrix takes the form

$$\begin{aligned} &S_{KK}^{KK}(\theta) = S_{\tilde{K}\tilde{K}}^{\tilde{K}\tilde{K}}(\theta) = S_{\tilde{K}K}^{\tilde{K}K}(\theta) = S_{K\tilde{K}}^{K\tilde{K}}(\theta) = 4i\sinh^2(2\theta)S_0(\theta) \\ &-S_{KK}^{\tilde{K}\tilde{K}}(\theta) = -S_{\tilde{K}\tilde{K}}^{KK}(\theta) = S_{K\tilde{K}}^{\tilde{K}K}(\theta) = S_{\tilde{K}K}^{K\tilde{K}}(\theta) = 4\sinh(2\theta)\sin\left(\frac{2\pi}{3}\right)S_0(\theta) \end{aligned} \tag{69}$$

Notice the sign flip in the second set of formulas. It will prove to be important below. In addition, S_0 now reads (choosing the negative sign in (22))

$$-\frac{1}{4i}\left(\sinh(2\theta)\sinh\left(2\theta - \frac{4\pi i}{3}\right)\right)^{-1}\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{5}{6}\right) \ . \tag{70}$$

The eigenvalues of the two-particle transition amplitudes turn out to be

$$-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{6}\right) \ , \ \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{5}{6}\right) \ , \tag{71}$$

however, now the first corresponds to the combinations

$$|K(\theta_1)\tilde{K}(\theta_2)\rangle - |\tilde{K}(\theta_1)K(\theta_2)\rangle \ , |K(\theta_1)K(\theta_2)\rangle + |\tilde{K}(\theta_1)\tilde{K}(\theta_2)\rangle \ , \tag{72}$$

while the second one to the states

$$|K(\theta_1)K(\theta_2)\rangle - |\tilde{K}(\theta_1)\tilde{K}(\theta_2)\rangle \ , \ |K(\theta_1)\tilde{K}(\theta_2)\rangle + |\tilde{K}(\theta_1)K(\theta_2)\rangle \ . \tag{73}$$

Due to the sign flip, each eigenvalue now corresponds to a symmetric and an antisymmetric combination. Note that the amplitudes in (71) are just crossing symmetric partners of each other. Therefore we are led to introduce the following particles:

$$|A\rangle = (|K\rangle + i|\tilde{K}\rangle)/\sqrt{2} \ , \ |\bar{A}\rangle = (|K\rangle - i|\tilde{K}\rangle)/\sqrt{2} \ , \tag{74}$$

so that the S-matrix takes the form

$$S_{AA} = S_{\bar{A}\bar{A}} = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{5}{6}\right) \ , \ S_{A\bar{A}} = -\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{6}\right) \ , \tag{75}$$

and we can treat A and $\bar{A}$ as conjugates of each other.

The unrestricted model contains a higher kink triplet and two breathers as well. The kink-higher kink and higher kink-higher kink S-matrices can be calculated using bootstrap for the bound state poles.

The higher kink pole is in the space spanned by the vectors in (72). We call L the higher kink coming from the channel given by the second vector and

$\tilde{L}$ the one coming from the first. Then the KL S-matrix turns out to be the following:

$$\begin{aligned}
&S_{KL}^{KL}(\theta) = S_{\tilde{K}\tilde{L}}^{\tilde{K}\tilde{L}}(\theta) = S_{\tilde{K}L}^{\tilde{K}L}(\theta) = S_{K\tilde{L}}^{K\tilde{L}}(\theta) = \\
&\frac{2(-1+i\sqrt{3})(y^2+1)}{(2y+\sqrt{3}+i)(2y-\sqrt{3}-i)} S_0^{KL}(\theta) \\
&-S_{KL}^{\tilde{K}\tilde{L}}(\theta) = -S_{\tilde{K}\tilde{L}}^{KL}(\theta) = S_{K\tilde{L}}^{\tilde{K}L}(\theta) = S_{\tilde{K}K}^{K\tilde{K}}(\theta) = \\
&-\frac{2(-1+i\sqrt{3})\sqrt{3}y}{(2y+\sqrt{3}+i)(2y-\sqrt{3}-i)} S_0^{KL}(\theta) \ ,
\end{aligned} \tag{76}$$

where

$$S_0^{KL}(\theta) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{12}\right)^2\left(\frac{7}{12}\right)\left(\frac{11}{12}\right) \ . \tag{77}$$

The S-matrix $S^{KL}(\theta)$ given in (76) is pseudounitary, i.e. satisfies

$$S^{KL}(-\theta) A S^{KL}(\theta) A = I \ , \tag{78}$$

where I is the 4×4 unit matrix and $A = \mathrm{diag}(1,1,-1,-1)$. Diagonalizing $S^{KL}(\theta)$ we find the eigenvalues

$$\begin{aligned}
S_1^{KL}(\theta) &= \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{12}\right)\left(\frac{11}{12}\right) \frac{\sinh\frac{1}{2}\left(\theta+\frac{5}{12}i\pi\right)\sinh\frac{1}{2}\left(\theta+\frac{7}{12}i\pi\right)}{\sinh\frac{1}{2}\left(\theta-\frac{1}{12}i\pi\right)\sinh\frac{1}{2}\left(\theta-\frac{11}{12}i\pi\right)} \ , \\
S_2^{KL}(\theta) &= \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{12}\right)\left(\frac{11}{12}\right) \frac{\sinh\frac{1}{2}\left(\theta+\frac{1}{12}i\pi\right)\sinh\frac{1}{2}\left(\theta+\frac{11}{12}i\pi\right)}{\sinh\frac{1}{2}\left(\theta-\frac{5}{12}i\pi\right)\sinh\frac{1}{2}\left(\theta-\frac{7}{12}i\pi\right)} \ .
\end{aligned} \tag{79}$$

The eigenvectors look very similar as in the case of the fundamental kink S-matrix. In the basis of the eigenvectors A takes the form

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ . \tag{80}$$

So pseudounitarity means that the eigenvalues are not pure phases but rather satisfy

$$S_1^{KL}(\theta) S_2^{KL}(-\theta) = 1 \ . \tag{81}$$

The appearance of such matrices A after RSOS restriction was already noticed in [3]. They are allowed by quantum group symmetry since it only tells us that states corresponding to different RSOS sequences must be orthogonal (cf. the discussion in section 4.1 and references therein), hence A should be diagonal in the basis of RSOS states, with its diagonal entries being ± 1 when properly normalized. A can be thought of as a metric on the state space (in this case the subspace of kink-higher kink two-particle states) and is (partially at least) fixed by the pseudounitarity requirement. It must coincide with the metric on the state space originating from the unperturbed CFT. Note that the eigenvectors of

$S^{KL}(\theta)$ are of zero pseudonorm with respect to A and also that the eigenvalues could have been obtained by applying diagonal bootstrap rules starting from the fundamental kink phaseshifts.

From (25) we learn that the higher kink-higher kink S-matrix is proportional to the fundamental kink S-matrix. Therefore one can introduce the states B and $\bar{B}$, following the analogy of A and $\bar{A}$. These states will then diagonalize the kink S-matrices. The remaining S-matrices present no additional novelties and are the following:

$$
\begin{aligned}
S_{BB} = S_{\bar{B}\bar{B}} &= \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 , \\
S_{B\bar{B}} &= -\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 , \\
S_{AC} = S_{\bar{A}C} &= \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{5}{12}\right)\left(\frac{7}{12}\right) , \\
S_{BC} = S_{\bar{B}C} &= \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 , \\
S_{CC} &= -\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{2}\right)^2 , \\
S_{AD} = S_{\bar{A}D} &= \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right)^2 , \\
S_{BD} = S_{\bar{B}D} &= \left(\frac{1}{12}\right)\left(\frac{11}{12}\right)\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^3\left(\frac{5}{12}\right)^4\left(\frac{7}{12}\right)^4 , \\
S_{CD} &= \left(\frac{1}{12}\right)\left(\frac{11}{12}\right)\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2\left(\frac{5}{12}\right)^3\left(\frac{7}{12}\right)^3 , \\
S_{DD} &= -\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^5\left(\frac{1}{2}\right)^6 .
\end{aligned}
\tag{82}
$$

The masses in the model are given by

$$
\begin{aligned}
m_A = m_{\bar{A}} = m,\ m_B = m_{\bar{B}} = 2m\cos\left(\frac{\pi}{12}\right), \\
m_C = 2m\cos\left(\frac{\pi}{4}\right),\ m_D = 4m\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{4}\right) .
\end{aligned}
\tag{83}
$$

The first is the fundamental kink mass, the next one is the mass of the higher kink and the last two are the two breathers.

The S-matrix has a Z_2 invariance, which exchanges A with $\bar{A}$ and B with $\bar{B}$, while it leaves C and D invariant. This is to be expected from the fact that the minimal model $\mathcal{M}_{(3,14)}$ (similarly to $\mathcal{M}_{(3,10)}$) has a Z_2 invariance and the perturbing operator is Z_2-even.

To establish consistency of the above picture, it is also useful to check the bootstrap consistency equations which constrain the higher spin conserved charges. For the E_6 case of the related unitary theory $\mathcal{M}_{(6,7)} + \Phi_{(1,2)}$, this has been done in [22]. The consistency equations given there can be easily changed to reflect the new fusion rules and they still allow for charges with spins

$$
s \equiv 1\ , 5 \bmod 6\ . \tag{84}
$$

These charges are Z_2-invariant, i.e., they take the same value on A and $\bar{A}$ and similarly for B and $\bar{B}$. This is just the Z_2-even subset of the charges allowed by the E_6 fusion rules. A further check of the above S-matrix is provided by the TCSA and TBA analysis given in [9].

Let me now make some remarks on the sign flip, observed in (69), which is characteristic for all $\mathcal{M}_{(3,4n+2)}$ models. The reason is that that the signs depend on the arithmetic properties of q. Generically, the models of the form $\mathcal{M}_{(3,4n+2)}$ are diagonalizable in terms of self-conjugate particles, if

$$4n+2 \equiv 10, 26 \text{ mod } 24 \tag{85}$$

(since the tensor part of the amplitude is the same as for $\mathcal{M}_{(3,10)}$, the only difference is in the definition of y in terms of θ and in the factor $S_0(\theta)$), and have flipped assignment of eigenvectors and therefore diagonalizable in terms of conjugate particle pairs if

$$4n+2 \equiv 14, 22 \text{ mod } 24 \ , \tag{86}$$

as can be proven by straightforward calculation of the amplitudes. The period 24 just reflects the periodic dependence of q on n. I would like to remark that even though the S-matrix is diagonalizable, the two particles are not independent, except in the case $\mathcal{M}_{(3,10)}$, since the amplitude S_{AB} is generically different from 1. The $\mathcal{M}_{(3,10)}$ case is special due to its tensor product form in terms of $\mathcal{M}_{(2,5)}$.

5.5 The model $\mathcal{M}_{(3,16)} + \Phi_{(1,5)}$

Let us close this section with the model $\mathcal{M}_{(3,16)} + \Phi_{(1,5)}$. The corresponding unitary model is $\mathcal{M}_{(3,4)} + \Phi_{(1,2)}$, the magnetic perturbation of the Ising model. In the Ising case $q = \exp(4i\pi/3)$, hence the allowed maximal spin is 1/2 and since the kinks are in the triplet representation of $\mathcal{A}_1$, all the kink degrees of freedom are frozen and only breathers remain in the spectrum. There are 8 breathers and the scattering is described by the so-called E_8 S-matrix [5].

However, in the new restriction, the kinks have singlet components so they can never be frozen. In addition, since $q' = \exp(16\pi i/3)$ gives $j_{max} = 1/2$, the charged kinks are allowed to remain in the spectrum as well. Therefore one can expect to see the masses of the kink and the higher kinks in the spectrum. This is different from the models $\mathcal{M}_{(3,10)} + \Phi_{(1,5)}$ and $\mathcal{M}_{(3,14)} + \Phi_{(1,5)}$, where the spectra (superficially at least) are the same as those of the corresponding $\Phi_{(1,2)}$ perturbed CFT. The term superficially indicates that the scattering of the particles is different in $\mathcal{M}_{(3,10)} + \Phi_{(1,5)}$ and $\mathcal{M}_{(3,14)} + \Phi_{(1,5)}$ from that of the related unitary models.

The S-matrix can be written in a very similar form as in the $\mathcal{M}_{(3,10)} + \Phi_{(1,5)}$ and $\mathcal{M}_{(3,14)} + \Phi_{(1,5)}$ case. Explicitly, it is given by:

$$S_{KK}^{KK} = \frac{y^4 - 2i\sqrt{3}y^2 - 1}{y^2} S_0(\theta) \ , \ S_{\tilde{K}\tilde{K}}^{\tilde{K}\tilde{K}} = -\frac{y^4 + 2i\sqrt{3}y^2 - 1}{y^2} S_0(\theta) \ ,$$
$$S_{\tilde{K}K}^{K\tilde{K}} = S_{K\tilde{K}}^{\tilde{K}K} = -i\sqrt{3}\frac{y^2+1}{y} S_0(\theta) \ , \ S_{\tilde{K}\tilde{K}}^{KK} = S_{KK}^{\tilde{K}\tilde{K}} = -\sqrt{3}\frac{y^2-1}{y} S_0(\theta) \ ,$$

$$S_{\tilde{K}K}^{\tilde{K}K} = S_{K\tilde{K}}^{K\tilde{K}} = \frac{y^4 - 1}{y^2} S_0(\theta) \ , \tag{87}$$

where

$$y = \exp\left(\frac{5\theta}{2}\right) \ , \tag{88}$$

and (choosing the positive sign in (22))

$$S_0(\theta) = \frac{1}{4i}\left(\sinh\frac{5}{2}(\theta - \pi i)\sinh\frac{5}{2}\left(\theta - \frac{2\pi i}{3}\right)\right)^{-1} \times \\ \exp\left(-2i\int_0^\infty \frac{\sin k\theta \sinh\frac{\pi k}{3}\cosh\frac{\pi}{30}k}{k\cosh\frac{\pi k}{2}\sinh\frac{\pi k}{5}}dk\right) \ , \tag{89}$$

since $\xi = 2\pi/5$.

However, the integral (89) cannot be carried out in closed form. In addition, it turns out, that while part of the S-matrix is diagonalizable on rapidity-independent combinations of the RSOS states, there is a part, which is not. Direct computations show that this is the case in all $\mathcal{M}_{(3,4n)}$ models. This phenomenon can be retraced to the fact that the arithmetical properties of q are different in the two class of models given by $\mathcal{M}_{(3,4n+2)}$ and $\mathcal{M}_{(3,4n)}$.

Let me spell out the eigenvalues, which is useful for the TCSA check [9]. The first two eigenvalues of the two-particle S-matrix are:

$$\frac{\sinh\left(\frac{5}{4}\theta - i\frac{\pi}{6}\right)}{\sinh\left(\frac{5}{4}\theta + i\frac{\pi}{6}\right)}\exp\left(-2i\int_0^\infty \frac{\sin k\theta \sinh\frac{\pi k}{3}\cosh\frac{\pi}{30}k}{k\cosh\frac{\pi k}{2}\sinh\frac{\pi k}{5}}dk\right) \ , \\ \frac{\sinh\left(\frac{5}{4}\theta + i\frac{\pi}{3}\right)}{\sinh\left(\frac{5}{4}\theta - i\frac{\pi}{3}\right)}\exp\left(-2i\int_0^\infty \frac{\sin k\theta \sinh\frac{\pi k}{3}\cosh\frac{\pi}{30}k}{k\cosh\frac{\pi k}{2}\sinh\frac{\pi k}{5}}dk\right) \ , \tag{90}$$

corresponding to the vectors

$$|K(\theta_1)\tilde{K}(\theta_2)\rangle + |\tilde{K}(\theta_1)K(\theta_2)\rangle \ , \ |K(\theta_1)\tilde{K}(\theta_2)\rangle - |\tilde{K}(\theta_1)K(\theta_2)\rangle \ , \tag{91}$$

respectively, while the second pair takes the more complicated form

$$\frac{1}{4}\frac{2i\sqrt{3} + 2\sinh\left(\frac{5}{2}\theta\right)\sqrt{2\cosh\left(5\theta\right) + 5}}{\cosh\frac{5}{2}\theta\sinh\left(\frac{5}{2}\theta - \frac{2\pi i}{3}\right)}\exp\left(-2i\int\dots\right) \ , \\ \frac{1}{4}\frac{2i\sqrt{3} - 2\sinh\left(\frac{5}{2}\theta\right)\sqrt{2\cosh\left(5\theta\right) + 5}}{\cosh\frac{5}{2}\theta\sinh\left(\frac{5}{2}\theta - \frac{2\pi i}{3}\right)}\exp\left(-2i\int\dots\right) \ , \tag{92}$$

(the dots denote the same integrand as above) and corresponds to the vectors

$$|K(\theta_1)K(\theta_2)\rangle + \frac{1}{\sqrt{3}}\left(2\cosh\frac{5}{2}\theta + \sqrt{2\cosh 5\theta + 5}\right)|\tilde{K}(\theta_1)\tilde{K}(\theta_2)\rangle \ , \\ |K(\theta_1)K(\theta_2)\rangle + \frac{1}{\sqrt{3}}\left(2\cosh\frac{5}{2}\theta - \sqrt{2\cosh 5\theta + 5}\right)|\tilde{K}(\theta_1)\tilde{K}(\theta_2)\rangle \ , \tag{93}$$

with $\theta = \theta_1 - \theta_2$ being the rapidity difference between the particles.

Due to the fact that the eigenvalues do not come in doubly degenerate pairs, in contrast to the case of the models $\mathcal{M}_{(3,4n+2)}$, there are no one-particle states on which the scattering can be diagonalized.

6 Conclusions

The results described above show that due to fact that the algebra $A_2^{(2)}$ is non-simply laced, the ZMS model allows another RSOS restriction, different from the one previuosly known. The S-matrices and the spectra of these theories can be derived by traditional methods of exact S-matrix theory and using the RSOS restriction procedure. The new restriction corresponds to the $\Phi_{(1,5)}$ perturbation of a different minimal model. The models $\mathcal{M}_{(3,10)}$, $\mathcal{M}_{(3,14)}$ and $\mathcal{M}_{(3,16)}$ have been investigated in detail. It is clear from the examples that the new RSOS restrictions have different S-matrices than the original $\Phi_{(1,2)}$ theory. In the $\mathcal{M}_{(3,16)} + \Phi_{(1,5)}$ case, even the mass spectrum is different. Therefore, even if one starts with the same unrestricted ZMS model, the two possible restrictions yield completely different physics.

As mentioned in the introduction, it would be of interest to extend the results of this paper to imaginary coupling affine Toda field theories based on more general non-simply laced affine Kac-Moody algebras, and to learn more about how the way, in which the RSOS restriction is performed, influences the physical picture obtained after the restriction.

The results presented here have been checked by applying the truncated conformal space approach combined with thermodynamical Bethe Ansatz calculations [9]. The results of the TCSA and TBA calculations are in complete accord with the details of the S-matrices and spectra described in this paper.

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